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EXPRESSION OF THE CO-EFFICIENTS OF STURM'S FUNCTIONS AS DETERMINANTS.

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Let $U = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots a_{n+1} = 0$ be an algebraic equation of the n^{th} degree,

$V = b_1x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \ldots b_n$ its first derived function, and

$$f_{m-2}(x) = c_1^{(m-2)}x^{n-(m-1)} + c_2^{(m-2)}x^{n-m} + c_3^{(m-2)}x^{n-(m+1)} + \ldots c_{n-(m-2)}^{(m-2)}, \quad (1)$$

$$f_{m-1}(x) = c_1^{(m-1)}x^{n-m} + c_2^{(m-1)}x^{n-(m+1)} + c_3^{(m-1)}x^{n-(m+2)} + \ldots c_{n-(m-1)}^{(m-1)}, \quad (2)$$

$$f_m(x) = c_1^{(m)}x^{n-(m+1)} + c_2^{(m)}x^{n-(m+2)} + c_3^{(m)}x^{n-(m+3)} + \ldots c_{n-m}^{(m)}, \quad (3)$$

the $(m-2)^{th}$, $(m-1)^{th}$, and m^{th} Sturmian functions. Multiply equation (2) by an arbitrary function $A_1x + A_2$, of the first degree in x, subtract equation (1), and determine A_1, A_2 so that the two highest terms shall vanish, and we get equation (3).

$$\therefore c_p^{(m)} = c_1^{(m-2)}c_1^{(m-1)}c_{p+2}^{(m-1)} - c_{p+2}^{(m-2)}\left[c_1^{(m-1)}\right]^2 + c_2^{(m-2)}c_1^{(m-1)}c_{p+1}^{(m-1)}$$
$$- c_1^{(m-2)}c_2^{(m-1)}c_{p+1}^{(m-1)}. \quad (4)$$

Equation (4) enables us to calculate $f_m(x)$ from $f_{m-2}(x)$ and $f_{m-1}(x)$.

We may express $c_p^{(m)}$ in the form of a determinant, and write

$$c_p^{(m)} = -\begin{vmatrix} c_1^{(m-2)}, & c_1^{(m-1)}, & 0 \\ c_2^{(m-2)}, & c_2^{(m-1)}, & c_1^{(m-1)} \\ c_{p+2}^{(m-2)}, & c_{p+2}^{(m-1)}, & c_{p+1}^{(m-1)} \end{vmatrix} = \begin{vmatrix} c_1^{(m-2)}, & 0, & c_1^{(m-1)} \\ c_2^{(m-2)}, & c_1^{(m-1)}, & c_2^{(m-1)} \\ c_{p+2}^{(m-2)}, & c_{p+1}^{(m-1)}, & c_{p+2}^{(m-1)} \end{vmatrix}. \quad (5)$$

The equality of the two expressions of $c_p^{(m)}$ may be verified by expanding equation (5), or we may deduce this equation at once, by observing that $c_p^{(m)}$ is the result of eliminating A_1, A_2 between the equations

$$-c_1^{(m-2)} + A_1c_1^{(m-1)} = 0,$$
$$-c_2^{(m-2)} + A_1c_2^{(m-1)} + A_2c_1^{(m-1)} = 0,$$
$$-c_{p+2}^{(m-2)} + A_1c_{p+2}^{(m-1)} + A_2c_{p+1}^{(m-1)} = 0.$$

In the case in which $f_{m-2}(x) = U$ and $f_{m-1}(x) = V$, we have

$$c_1^{(m-2)} = a_1, \qquad c_2^{(m-2)} = a_2, \qquad c_{p+2}^{(m-2)} = a_{p+2},$$
$$c_1^{(m-1)} = b_1 = na_1, \quad c_2^{(m-1)} = b_2 = (n-1)a_2, \quad c_{p+2}^{(m-1)} = b_{p+2} = (n-p-1)a_{p+2}.$$

$$\therefore c_p^{(1)} = -\begin{vmatrix} a_1, & b_1, & 0 \\ a_2, & b_2, & b_1 \\ a_{p+2}, & b_{p+2}, & b_{p+1} \end{vmatrix}. \quad (6)$$

Multiply the first column by n and subtract from the second, and we have

$$c_p^{(1)} = -\begin{vmatrix} a_1, & 0, & 0 \\ a_2, & -a_2, & b_1 \\ a_{p+2}, & -(p+1)\,a_{p+2}, & b_{p+1} \end{vmatrix}. \tag{7}$$

Hence, rejecting the factor a_1, supposed positive, we may write

$$c_p^{(1)} = \begin{vmatrix} a_2, & b_1 \\ (p+1)\,a_{p+2}, & b_{p+1} \end{vmatrix}. \tag{8}$$

It will, in general, be most convenient to deduce each of Sturm's functions from the two preceding by the aid of equations (3) and (5), but it may be desirable to express each function in terms of U and V; this may be done as follows:—

$$\begin{aligned} f_1(x) &= Q_1 V - U, \\ f_2(x) &= Q_2 f_1(x) - V \\ &= Q_1 Q_2 V - Q_2 U - V \\ &= (Q_1 Q_2 - 1)\,V - Q_2 U, \\ f_3(x) &= Q_3 f_2(x) - f_1(x) \\ &= Q_3 (Q_1 Q_2 - 1)\,V - Q_2 Q_3 V - Q_1 V + U \\ &= (Q_1 Q_2 Q_3 - Q_1 - Q_3)\,V - (Q_2 Q_3 - 1)\,U, \end{aligned}$$

where Q_1, Q_2, Q_3 are functions of the first degree in x.

In general
$$f_m(x) = P_m V - R_m U, \tag{9}$$

where P_m, R_m are functions of x of degree m, $m - 1$ respectively.

If
$$\begin{aligned} P_m &= A_1 x^m + A_2 x^{m-1} + A_3 x^{m-2} + \ldots A_{m+1}, \\ R_m &= B_1 x^{m-1} + B_2 x^{m-2} + B_3 x^{m-3} + \ldots B_m, \end{aligned}$$

we seem to have $2m + 1$ unknown quantities, but really have one less, since any one of the quantities A, B may be made $= 1$ by division; and, since we have to make the first $2m$ terms of equation (9) vanish, we have the right number of equations to enable us to determine the values of $A_1, A_2, A_3, \ldots A_{m+1}, B_1, B_2, B_3, \ldots B_m$. Expanding equation (9), we see that $c_p^{(m)}$ is the result of eliminating $A_1, A_2, A_3, \ldots A_{m+1}, B_1, B_2, B_3, \ldots B_m$ between the equations

$$\begin{array}{ll} -B_1 a_1 & +A_1 b_1 \\ -B_1 a_2 - B_2 a_1 & +A_1 b_2 + A_2 b_1 \\ -B_1 a_3 - B_2 a_2 - B_3 a_1 & +A_1 b_3 + A_2 b_2 + A_3 b_1 \\ -B_1 a_4 - B_2 a_3 - B_3 a_2 - B_4 a_1 & +A_1 b_4 + A_2 b_3 + A_3 b_2 + A_4 b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ -B_1 a_m - B_2 a_{m-1} - B_3 a_{m-2} - B_4 a_{m-3} \ldots & -B_m a_1 \\ & \quad +A_1 b_m + A_2 b_{m-1} \ldots + A_m b_1 \end{array}$$

$$\begin{aligned}&-B_1a_{m+1}-B_2a_m-B_3a_{m-1}-B_4a_{m-2}\;.\;.\;.-B_ma_2\\&\qquad\qquad+A_1b_{m+1}+A_2b_m\;.\;.\;.+A_mb_2-A_{m+1}\,b_1\end{aligned}$$

$$\begin{aligned}&-B_1a_{m+2}-B_2a_{m+1}-B_3a_m-B_4a_{m-1}\;.\;.\;.-B_ma_3\\&\qquad\qquad+A_1b_{m+2}+A_2b_{m+1}\;.\;.\;.+A_mb_3-A_{m+1}\,b_2\end{aligned}$$

. . . .

$$\begin{aligned}&-B_1a_{2m-1}-B_2a_{2m-2}-B_3a_{2m-3}-B_4a_{2m-4}\;.\;.\;.-B_ma_m\\&\qquad\qquad+A_1b_{2m-1}+A_2b_{2m-2}\;.\;.\;.+A_mb_m-A_{m+1}\,b_{m-1}\end{aligned}$$

$$\begin{aligned}&-B_1a_{2m}-B_2a_{2m-1}-B_3a_{2m-2}-B_4a_{2m-3}\;.\;.\;.-B_ma_{m+1}\\&\qquad\qquad+A_1b_{2m}+A_2b_{2m-1}\;.\;.\;.+A_mb_{m+1}-A_{m+1}b_m\end{aligned}$$

$$\begin{aligned}&-B_1a_{2m+p}-B_2a_{2m+p-1}-B_3a_{2m+p-2}-B_4a_{2m+p-3}\;.\;.\;.-B_ma_{m+p+1}\\&\qquad\qquad+A_1b_{2m+p}+A_2b_{2m+p-1}\;.\;.\;.+A_mb_{m+p+1}-A_{m+1}\,b_{m+p}\end{aligned}$$

$$\therefore\ c_p^{(m)} = \left|\begin{array}{llllllllll}
-a_1, & 0, & 0, & 0 & \ldots & 0, & b_1, & 0, & \ldots 0, & 0 \\
-a_2, & -a_1, & 0, & 0, & \ldots & 0, & b_2, & b_1, & \ldots 0, & 0 \\
-a_3, & -a_2, & -a_1, & 0, & \ldots & 0, & b_3, & b_2, & \ldots 0, & 0 \\
-a_4, & -a_3, & -a_2, & -a_1, & \ldots & 0, & b_4, & b_3, & \ldots 0, & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-a_m, & -a_{m-1}, & -a_{m-2}, & -a_{m-3}, & \ldots & -a_1, & b_m, & b_{m-1}, & \ldots b_1, & 0 \\
-a_{m+1}, & -a_m, & -a_{m-1}, & -a_{m-2}, & \ldots & -a_2, & b_{m+1}, & b_m, & \ldots b_2, & -b_1 \\
-a_{m+2}, & -a_{m+1}, & -a_m, & -a_{m-1}, & \ldots & -a_3, & b_{m+2}, & b_{m+1}, & \ldots b_3, & -b_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-a_{2m-1}, & -a_{2m-2}, & -a_{2m-3}, & -a_{2m-4}, & \ldots & -a_m, & b_{2m-1}, & b_{2m-2}, & \ldots b_m, & -b_{m-1} \\
-a_{2m}, & -a_{2m-1}, & -a_{2m-2}, & -a_{2m-3}, & \ldots & -a_{m+1}, & b_{2m}, & b_{2m-1}, & \ldots b_{m+1}, & -b_m \\
-a_{2m+p}, & -a_{2m+p-1}, & -a_{2m+p-2}, & -a_{2m+p-3}, & \ldots & -a_{m+p+1}, & b_{2m+p}, & b_{2m+p-1}, & \ldots b_{m+p+1}, & -b_{m+p}
\end{array}\right| . \quad (10)$$

Add n times the first column to the $(m+1)^{th}$, n times the second column to the $(m+2)^{th}$, and so on, observing that $a_k = -k + 1) b_k$, reject the factor a_1, and we have

$$c_p^{(m)} = -\left|\begin{array}{llllllll}
a_1, & 0, & 0, & \ldots 0, & a_2, & 0, & \ldots & 0, \quad 0 \\
a_2, & a_1, & 0, & \ldots 0, & 2a_3, & a_2, & \ldots & 0, \quad 0 \\
a_3, & a_2, & a_1, & \ldots 0, & 3a_4, & 2a_3, & \ldots & 0, \quad 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{m-1}, & a_{m-2}, & a_{m-3}, & \ldots a_1, & (m-1)\,a_m, & (m-2)\,a_{m-1}, & \ldots & 0, \quad 0 \\
a_m, & a_{m-1}, & a_{m-2}, & \ldots a_2, & m a_{m+1}, & (m-1)\,a_m, & \ldots & a_2, \quad b_1 \\
a_{m+1}, & a_m, & a_{m-1}, & \ldots a_3, & (m+1)\,a_{m+2}, & m a_{m+1}, & \ldots & 2a_3, \quad b_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{2m-2}, & a_{2m-3}, & a_{2m-4}, & \ldots a_m, & (2m-2)\,a_{2m-1}, & (2m-3)\,a_{2m-2}, & \ldots & (m-1)\,a_m, \quad b_{m-1} \\
a_{2m-1}, & a_{2m-2}, & a_{2m-3}, & \ldots a_{m+1}, & (2m-1)\,a_{2m}, & (2m-2)\,a_{2m-1}, & \ldots & m a_{m+1}, \quad b_m \\
a_{2m+p-1}, & a_{2m+p-2}, & a_{2m+p-3}, & \ldots a_{m+p+1}, & (2m+p-1)\,a_{2m+p}, & (2m+p-2)\,a_{2m+p-1}, & \ldots & (m+p)\,a_{m+p+1}, \quad b_{m+p}
\end{array}\right| . \quad (11)$$

EXAMPLE :— Let $U = x^4 - 8x^3 + 14x^2 + 4x - 8 = 0,$
$V = x^3 - 6x^2 + 7x + 1.$

By equation (8)

$$c_1^{(1)} = \begin{vmatrix} -8, & 1 \\ 28, & -6 \end{vmatrix} = 20,$$

$$c_2^{(1)} = \begin{vmatrix} -8, & 1 \\ 12, & 7 \end{vmatrix} = -68,$$

$$c_3^{(1)} = \begin{vmatrix} -8, & 1 \\ -32, & 1 \end{vmatrix} = 24;$$

$$\therefore f_1(x) = 5x^2 - 17x + 6.$$

By equation (5)

$$c_1^{(2)} = \begin{vmatrix} 1, & 0, & 5 \\ -6, & 5, & -17 \\ 7, & -17, & 6 \end{vmatrix} = 76,$$

$$c_2^{(2)} = \begin{vmatrix} 1, & 0, & 5 \\ -6, & 5, & -17 \\ 1, & 6, & 0 \end{vmatrix} = -103;$$

$$\therefore f_2(x) = 76x - 103.$$

$$f_3(x) = c_1^{(3)} = \begin{vmatrix} 5, & 0, & 76 \\ -17, & 76, & -103 \\ 6, & -103, & 0 \end{vmatrix} = 45375.$$

To calculate $f_2(x)$ and $f_3(x)$ by equation (11) we have

$$c_1^{(2)} = -\begin{vmatrix} 1, & -8, & 0, & 0 \\ -8, & 28, & -8, & 1 \\ 14, & 12, & 28, & -6 \\ 4, & -32, & 12, & 7 \end{vmatrix} = 16 \times 76,$$

$$c_2^{(2)} = -\begin{vmatrix} 1, & -8, & 0, & 0 \\ -8, & 28, & -8, & 1 \\ 14, & 12, & 28, & -6 \\ -8, & 0, & -32, & 1 \end{vmatrix} = -16 \times 103,$$

$$c_1^{(3)} = -\begin{vmatrix} 1, & 0, & -8, & 0, & 0, & 0 \\ -8, & 1, & 28, & -8, & 0, & 0 \\ 14, & -8, & 12, & 28, & -8, & 1 \\ 4, & 14, & -32, & 12, & 28, & -6 \\ -8, & 4, & 0, & -32, & 12, & 7 \\ 0, & -8, & 0, & 0, & -32, & 1 \end{vmatrix} = 64 \times 1815.$$